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Theoretical Basis and Application of an Analogue-Dynamical Model in the Lorenz System

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ABSTRACT

The theoretical basis and application of an analogue-dynamical model (ADM) in the Lorenz system is studied. The ADM can effectively combine statistical and dynamical methods in which the small disturbance of the current initial value superimposed on the historical analogue reference state can be regarded as a prediction objective. Primary analyses show that under the condition of appending disturbances in model parameters, the model errors of ADM are much smaller than those of the pure dynamical model (PDM).

The characteristics of predictability on the ADM in the Lorenz system are analyzed in phase space by conducting case studies and global experiments. The results show that the ADM can quite effectively reduce prediction errors and prolong the valid time of the prediction in most situations in contrast to the PDM, but when model errors are considerably small, the latter will be superior to the former. To overcome such a problem, the multi-reference-state updating can be applied to introduce the information of multi-analogue and update analogue and can exhibit exciting performance in the ADM.

Key words: analogue-dynamical model, Lorenz system, predictability, model errors

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1. Introduction

During recent decades, meteorologists have made great achievements in dynamical prediction based on numerical models (Kalnay, 2003; Mu et al., 2002). By following continual improvements of data and models, numerical weather forecasting and dynamical short-term climate prediction play an increasingly important role in routine operations. However, the applications of numerical prediction products in practice are still unsatisfactory under the current conditions, and the empirical and statistical method still has considerable prediction skill (van den Dool, 2007). Thus, in order to promote prediction performance by using existing data and models, prediction strategy and methodology based on numerical models have been proposed and have become an important approach for improving prediction (e. g., Hamill and Whitaker, 2006; see also review of Ren and Chou, 2007a). Many related researches were carried out, in which Chinese scholars have made significant contributions.

Early in the 1950s, Gu (1958) pointed out the significance and feasibility of introducing historical data into numerical forecast. Thereafter, a series of innovative prediction methods using historical data have been put forward and widely applied to prediction experiments, which exhibited considerable capability in improving prediction (e.g., Chou, 1974; Cao, 1993; Qiu and Chou, 1989; Huang and Wang, 1992; Huang et al., 1993; Gu, 1998; Gong et al., 1999; Feng et al., 2001; Bao et al., 2004; Ren and Chou, 2006a,b, 2007b; Gao et al., 2006; Ren et al., 2006; Ren, 2008). In these works, Chou (1979) suggested, for the first time, the essential idea of an analogue-dynamical approach (ADA) in which the predicted dynamical field is regarded as the small disturbance superimposed on the historical analogue field so that the synoptic experiences can be introduced to the numerical forecast. According to such an idea, some analogue-dynamical models (ADMs) were established based on the analogue-

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deviation versions of simple quasi-geostrophic models (Qiu and Chou, 1989; Huang and Wang, 1992; Huang et al., 1993).

Preliminary analyses show that the ADM has a greater accuracy than the pure dynamical model (PDM) in virtue of the compensating effect from historical analogue to model errors. Further, the ADA was applied to the monthly dynamical extended-range forecast (DERF) model and an equivalent ADM (Bao et al., 2004) essentially based on a method of analogue correction of errors (ACE) (Ren and Chou, 2006a; Gao et al., 2006) was established. Besides this, to introduce the information of multi-analogue and update selected historical analogue in the process of integrating the ADM, a method based on multi-reference-state updating (MRSU) has been recently put forward (Ren and Chou, 2006b; Ren et al., 2006). Related experiments for monthly DERF combining an equivalent ADM and the MSRU displayed some encouraging preliminary results.

Up to now, many important issues were unsolved in previous works and need to be further studied. On one hand, for both quasi-geostrophic and operational models, the predictability problems of corresponding ADMs need be deeply examined and more prediction experiments are very necessary for validating the applicability of the ADMs. On the other hand, many impact factors associated with model errors, such as error type, error magnitude, etc., have significant effects on the ADM prediction, which are worth being studied in detail. Moreover, how to better introduce historical analogue information into the ADM should also be of concern. All of these issues should be seriously studied from both theory analyses and numerical simulation. However, it is quite difficult for these real atmospheric ADMs to conduct comprehensive experiments under the conditions of various predictabilities. In this paper, the ADM in the Lorenz system will be used to explore the above-mentioned problems. The characteristics of predictability on the ADM and the combination of it with the MRSU are further examined in phase space by conducting case studies and global experiments.

2. Analogue-dynamical approach

In general, numerical prediction models can be expressed as

$$\frac{\partial \boldsymbol{\psi}}{\partial t} + L(\boldsymbol{\psi}) = 0 \,, \tag{1}$$

$$\boldsymbol{\psi}(\boldsymbol{r}, t_0) = G(\boldsymbol{r}) \,, \tag{2}$$

where $\boldsymbol{\psi}(\boldsymbol{r}, t)$ is the model state vector to be predicted, $\boldsymbol{r}$ is the vector in the spatial coordinates, t_0 is initial time, L is the differential operator of $\boldsymbol{\psi}$, which is usually nonlinear and corresponding to real numerical model. Similarly, the exact model satisfied by the real atmosphere can be expressed as

$$\frac{\partial \boldsymbol{\psi}}{\partial t} + L(\boldsymbol{\psi}) = E(\boldsymbol{\psi}) \,, \tag{3}$$

where E is the error operator which stands for the process that exists in reality but is not described exactly in Eq. (1), and reflects the error term of real numerical model. Historical data may then be naturally regarded as a series of special solutions or their functions of Eqs. (3) and (2).

According to basic idea of the analogue-dynamical approach (Chou, 1979; Qiu and Chou, 1989; Huang et al., 1993), $\boldsymbol{\psi}$ can be divided into the analogue reference state (or reference state for short, denoted as RS) $\tilde{\boldsymbol{\psi}}$ and the analogue disturbance state (or disturbance state for short, denoted DS) $\boldsymbol{\psi}'$. Thus, we have $\boldsymbol{\psi} = \tilde{\boldsymbol{\psi}} + \boldsymbol{\psi}'$, where $\tilde{\boldsymbol{\psi}}$ is selected from historical observations in terms of the similarities between the RSs and current initial value $G(\boldsymbol{r})$. The reference state satisfies the following equations:

$$\frac{\partial \tilde{\boldsymbol{\psi}}}{\partial t} + L(\tilde{\boldsymbol{\psi}}) = E(\tilde{\boldsymbol{\psi}}) \,, \tag{4}$$

$$\tilde{\boldsymbol{\psi}}(\boldsymbol{r}, t_{\rm h}) = \tilde{G}(\boldsymbol{r}) \,, \tag{5}$$

where $t_{\rm h}$ is historical time. By subtracting Eqs. (4) and (5) from Eqs. (3) and (2) respectively, the exact equation satisfied by the DS is obtained as follows:

$$\frac{\partial \boldsymbol{\psi}'}{\partial t} + L(\tilde{\boldsymbol{\psi}} + \boldsymbol{\psi}') - L(\tilde{\boldsymbol{\psi}}) = E(\tilde{\boldsymbol{\psi}} + \boldsymbol{\psi}') - E(\tilde{\boldsymbol{\psi}}) \,, \tag{6}$$

$$\boldsymbol{\psi}'(\boldsymbol{r}, t_0) = G(\boldsymbol{r}) - \tilde{G}(\boldsymbol{r}) \,. \tag{7}$$

Similarly, substituting $\boldsymbol{\psi} = \tilde{\boldsymbol{\psi}} + \boldsymbol{\psi}'$ and $\tilde{\boldsymbol{\psi}}$ into Eq. (1) respectively, and subtracting the latter from the former, we obtain the analogue-deviation equation:

$$\frac{\partial \boldsymbol{\psi}'}{\partial t} + L(\tilde{\boldsymbol{\psi}} + \boldsymbol{\psi}') - L(\tilde{\boldsymbol{\psi}}) = 0 \,. \tag{8}$$

By first selecting historical analogue $\tilde{\boldsymbol{\psi}}$ of the current initial state as the RS, the DS $\boldsymbol{\psi}'$ is calculated in terms of Eqs. (8) and (7), and current prediction $\boldsymbol{\psi}$ is obtained by $\boldsymbol{\psi} = \tilde{\boldsymbol{\psi}} + \boldsymbol{\psi}'$. Eqs. (8) and (6) are corresponding to the inexact numerical model and the exact model satisfied by the real atmosphere, respectively. Also, Eq. (8) can be obtained by omitting $E(\tilde{\boldsymbol{\psi}} + \boldsymbol{\psi}') - E(\tilde{\boldsymbol{\psi}})$ on the right side of Eq. (6), which is evidently more precise than the way that Eq. (1) is obtained by omitting $E(\boldsymbol{\psi})$ on the right side of Eq. (3). So the ADM on the basis of Eq. (8) has much fewer model errors than the ordinary model on the basis of Eq. (1). This suggests that such an ADM will have greater accuracy than either the pure dynamical model (PDM) or the statistical analogue prediction (Barnett and Preisendorfer, 1978; van den Dool, 1987; Toth, 1989; Livezey et al., 1994), owing to the compensation effect from historical analogues to model errors, which has been documented by the monthly and seasonal prediction experiments (Qiu and Chou, 1989; Huang and Wang, 1992; Huang et al., 1993; Bao et al., 2004).

3. PDM and ADM in the Lorenz system

The control equation group of the Lorenz system (Lorenz, 1963), also called the Lorenz model, which stands for a classical nonlinear chaotic system, is expressed as follows:

$$\begin{cases} \dfrac{dx}{dt} = -\sigma x + \sigma y \ , \\ \dfrac{dy}{dt} = -xz + rx - y \ , \\ \dfrac{dz}{dt} = xy - bz \ , \end{cases} \tag{9}$$

where, $\sigma, \boldsymbol{r}$ and b are model parameters. Take $\sigma = 10, \boldsymbol{r} = 28$ and $b = 8/3$, respectively. The 4-order Runge-Kuta integration algorithm is employed for solving ordinary differential equation groups in this paper and the integration time step is taken as $\Delta t = 0.01$.

As well known, the Lorenz model has been widely used in theoretical studies of nonlinear dynamics and predictability (e.g., Chou, 1995; Mu et al., 2002). In this paper, in order to generate historical data similar to the real atmosphere, the exact Lorenz model in Eq. (9) is integrated for 257500 time steps. After removing the initial 2000-step data during the adaptation period, the simulated dataset, in chaotic situation, is regarded as the observed dataset in the Lorenz system, where observation errors are omitted. Furthermore, provided that 20 time steps correspond to 1 day in practice, the above-described observed dataset will cover 35 years ×365 days. We take initial 30-year data as historical data for selecting analogue, and residual 5-year data as independent samples for verifying predictions.

As the approximation of the real atmosphere, numerical models inevitably have more or less model errors. If we regard the Lorenz system as the simplest atmosphere, similar to real numerical models, the Lorenz model with model errors may be expressed as

$$\begin{cases} \dfrac{dx}{dt} = -\sigma x + \sigma y + e_x \ , \\ \dfrac{dy}{dt} = -xz + rx - y + e_y \ , \\ \dfrac{dz}{dt} = xy - bz + e_z \ , \end{cases} \tag{10}$$

where $(e_x, e_y, e_z)^{\mathrm{T}}$ is model error vector. In the following sections, Eq. (10) is also called the pure dynamical model (PDM) of the Lorenz system.

Let $x = \tilde{x} + x', y = \tilde{y} + y'$ and $z = \tilde{z} + z'$, in which $\tilde{x}, \tilde{y}$ and $\tilde{z}$ are RSs, and x', y', and z' are DSs. In terms of the derivation process described in Eqs. (1)–(8), by substituting x, y, z and $\tilde{x}, \tilde{y}, \tilde{z}$ into Eq. (10) respectively, and subtracting the latter from the former, the ADM corresponding Eq. (10) with model errors may be obtained as follows:

$$\begin{cases} \dfrac{dx'}{dt} = -\sigma x' + \sigma y' + e'_x \ , \\ \dfrac{dy'}{dt} = -\tilde{x}z' - \tilde{z}x' - x'z' + rx' - y' + e'_y \ , \\ \dfrac{dz'}{dt} = \tilde{x}y' + \tilde{y}x' + x'y' - bz' + e'_z \ . \end{cases} \tag{11}$$

Here, $(e'_x, e'_y, e'_z)^{\mathrm{T}}$ stands for the model error vector of the ADM.

4. Theoretical analysis

The basic idea of the ADA still needs to be documented by further theoretical analyses. More prediction experiments are also very necessary for examining the applicability of the ADMs in practice. However, it is quite difficult for existing real atmospheric ADMs to conduct comprehensive experiments under the conditions of various predictabilities. As a common study tool, the Lorenz system is more suitable for theoretical analyses and comprehensive experiments, which has a good use as a reference for complex numerical models. Objectively speaking, the truth does not vary with the number of degrees of freedom in a dynamical system.

As we know, forecast errors are evidently related to errors in initial conditions and model errors, where the former is well concerned in previous works, and many ideas have been proposed for improving forecast skill (e.g., Pu et al., 1997a,b; Mu and Wang, 2001; Duan and Mu, 2005; Mu and Jiang, 2007; Mu et al., 2007). Comparatively, studies associated with model errors are still few (e.g., Orrell, 2003). Considering that forecast errors generated from model errors have a significant effect on dynamical prediction skill, it is necessary to theoretically learn about the model error vector represented by $(e_x, e_y, e_z)^{\mathrm{T}}$. More concretely, we may take different types of model errors and conduct theoretical analyses and numerical experiments by employing the ADM to examine whether it can effectively reduce the given model errors. First of all, we will identify the classification of model errors.

4.1 *Classification of model errors*

In general, model errors $E_{\rm model}$ may be divided into the following three parts:

$$E_{\rm model} = E_{\rm sys} + E_{\rm flow} + E_{\rm stoch}\ .$$

$E_{\rm sys}$ is systematic error and also called as climatic drift. $E_{\rm flow}$ may be named as time-dependent or flow-dependent error and is the model error varying with system state or flow regime. $E_{\rm stoch}$ is stochastic error and can not generally be overcome, and is not considered here. At present, based on theoretical and experimental researches, the purpose of developing numerical models is to simply reduce model errors by improving dynamical frameworks, physical processes, and so on. But regardless, there always objectively exist considerable errors in models. Consequently, according to the idea of studying prediction strategy (Ren and Chou, 2007a), model errors can be estimated and reduced by utilizing historical data information based on inverse problems under the condition of existing models (Ren and Chou, 2006a, 2007b; Gao et al., 2006).

4.2 *Theoretical analyses of model errors*

In the following, we will theoretically discuss the performance of the ADM in reducing model errors based on historical analogue.

4.2.1 *Systematic error*

For $E_{\rm sys}$, it can be overcome by developing a deviation model and can also be indirectly removed by eliminating model climatology from prediction results. Then what about the ADM? Here, we take the error vector of the Lorenz prediction model $(e_x, e_y, e_z)^{\rm T} = (s_1, s_2, s_3)^{\rm T}$ as the systematic model error, where s_1, s_2, s_3 are constants. It is easily seen that only for $E_{\rm sys}$, the model error vector in Eq. (11) $(e'_x, e'_y, e'_z)^{\rm T} = (0, 0, 0)^{\rm T}$, which shows that there is no $E_{\rm sys}$ in the ADM. In other words, the ADM can completely eliminate such $E_{\rm sys}$, which need not be documented by experiments.

4.2.2 *Time-dependent error*

For $E_{\rm flow}$, it varies with flow regime and is the function of model states or variables. Similar to model error types in really complicated numerical models, in the Lorenz prediction model $E_{\rm flow}$ can be suitably represented by appending small disturbances on the three model parameters in Eq. (9). In the following, we will respectively discuss $E_{\rm flow}$ according to different given situations. First supposed that

$$\parallel x' \parallel < \parallel x \parallel\ , \parallel y' \parallel < \parallel y \parallel\ , \parallel z' \parallel < \parallel z \parallel\ ,$$

where, the norm $\parallel \cdot \parallel$ of metric is taken as the inner product of two vectors.

(1) Append the disturbance $\delta \boldsymbol{r}$ on parameter $\boldsymbol{r}$. At this time, the model error vector in Eq. (10) $(e_x, e_y, e_z)^{\rm T} = (0, \delta rx, 0)^{\rm T}$, and then that of the ADM in Eq. (11) is $(0,\ \delta rx', 0)^{\rm T}$. Consequently, we can easily obtain the relationship between the two model error metrics as follows:

$$\parallel \delta rx' \parallel < \parallel \delta rx \parallel\ .$$

(2) Append the disturbance δb on parameter b. The model error vectors in Eqs. (10) and (11) are $(0, 0, -\delta bz)^{\rm T}$ and $(0, 0, -\delta bz')^{\rm T}$, respectively. We can easily obtain

$$\parallel -\delta bz' \parallel < \parallel -\delta bz \parallel\ .$$

(3) Append the disturbance $\delta\sigma$ on parameter σ. The model error vectors in Eqs. (10) and (11) are $(-\delta\sigma x + \delta\sigma y, 0, 0)^{\rm T}$ and $(-\delta\sigma x' + \delta\sigma y', 0, 0)^{\rm T}$, respectively. Here, by introducing $w = y - x$, have $w = \tilde{w} + w'$ and suppose $\parallel w' \parallel < \parallel w \parallel$, we can easily obtain

$$\parallel \delta\sigma w' \parallel < \parallel \delta\sigma w \parallel\ .$$

As above, it has been theoretically documented that some representative $E_{\rm flow}$ induced by model parameter errors in the Lorenz model can be effectively overcome in the ADM by utilizing historical analogue information.

4.2.3 *General situation*

For more general situations, similar to the model errors $E(\boldsymbol{\psi})$ in Eq. (3), model errors of the Lorenz model may be expressed as $E(x, y, z)$, then those of corresponding historical analogue are $E(\tilde{x}, \tilde{y}, \tilde{z})$. Then, Taylor-expand $E(x, y, z)$ to first order around $(\tilde{x}, \tilde{y}, \tilde{z})$ as follows:

$$E(x, y, z) = E(\tilde{x} + x', \tilde{y} + y', \tilde{z} + z')$$
$$\equiv E(\tilde{x}, \tilde{y}, \tilde{z}) + x'\frac{\partial E}{\partial x}\bigg|_{\tilde{x}} + y'\frac{\partial E}{\partial y}\bigg|_{\tilde{y}} + z'\frac{\partial E}{\partial z}\bigg|_{\tilde{z}}\ .$$

As we can see, when the partial differential of E with respect to every component is bounded respectively and $\parallel (x', y', z') \parallel$ is small enough, it is not difficult to obtain

$$\parallel E(x, y, z) - E(\tilde{x}, \tilde{y}, \tilde{z}) \parallel \ll \parallel E(x, y, z) \parallel\ .$$

This suggests that the ADM of the Lorenz system on the basis of Eq. (11) has fewer model errors and higher accuracy than the PDM on Eq. (10) under given conditions, although the two kinds of models are both inexact. Thus, by selecting the RS $(\tilde{x}, \tilde{y}, \tilde{z})$ of current initial value first, the DS (x', y', z') can be calculated in terms of Eq. (11) and the current forecast (x, y, z) can be finally obtained by $x = \tilde{x} + x'$, $y = \tilde{y} + y'$ and $z = \tilde{z} + z'$.

5. Case study

Firstly, we will exhibit a detailed prediction process and results based on the ADM of the Lorenz system by conducting a case study.

5.1 *Analogue metric and verification statistics*

Analogue metric for analogue selection is defined as a simple Euclid distance function:

$$d = [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{1/2} ,$$

where suffixes 1 and 2 denote any two state points respectively. The smaller d between two states is, the more similar they are.

It is easily understood that the verification problem of prediction results of realistic experiments based the Lorenz model is clearly different from that of real atmospheric model. The latter is focused on reducing field prediction errors and improving pattern prediction skill, whereas the former is more focused on prolonging valid time of prediction. Many experiments have shown that prediction errors will suddenly increase to no skill as long as the transition between two equilibriums is predicted by mistake. To conduct global predictability analyses, we utilize a relative prediction error defined as follows:

$$l(\tau) = \left\{ \frac{[x_{\rm f}(\tau) - x_{\rm o}(\tau)]^2 + [y_{\rm f}(\tau) - y_{\rm o}(\tau)]^2 + [z_{\rm f}(\tau) - z_{\rm o}(\tau)]^2}{[x_{\rm o}(\tau)]^2 + [y_{\rm o}(\tau)]^2 + [z_{\rm o}(\tau)]^2} \right\}^{1/2} ,$$

where τ is lead time of prediction, suffixes f and o stand for forecast and observe respectively.

As we have known, there is little bias between prediction and verification before some lead time. But after this time, the forecast will deviate from the observation and l will increase quickly whether to integrate the Lorenz model without model errors by using a set of initial values with small errors or with model errors by using accurate initial values. Based on many experiments, we find once prediction becomes unacceptable, the corresponding time is defined as the valid time (denoted as VT) of prediction. Here, the VT may give prediction limit and prediction will lose all of skill after the VT. In the present paper, we take the VT as an objective metric for dynamical predictive effect of the Lorenz model.

5.2 *Result analysis*

Here, we take a representative case as visual analysis under the condition of $\delta \boldsymbol{r} = 0.1$ with initial value $(-6.497, -0.5085, 31.59)$ reserved from 15 to 4 significant digits. The best analogue selected by distance in phase space is regarded as the reference state of the ADM and the interval of observed sample (IOS) is taken as 5 time steps. Figure 1 gives predicted and verified x curves under the condition of given model parameter error based on the PDM and ADM respectively.

We may easily obtain the VT in Figs. 1a and b with 4.97 and 5.77 respectively, which shows that in the situation of no observation errors, dynamical predictions are significantly affected by model parameter error. It can be clearly seen from Fig. 1 that before the VT, the predicted and verified x curves are very close to each other whether based on the PDM or the ADM, but after the VT, they quickly separate and enter different equilibriums when predictions appear unacceptable. Comparatively, according to this case, the ADM with longer VT can exactly predict two more transitions between two equilibriums and exhibit better performance than the PDM, which will be further documented in the following global analysis.

6. ADM global experiments

In order to comprehensively examine the performance of the ADM under different conditions of predictability in phase space, we conduct global prediction experiments based on the ADM according to the disturbance schemes of model parameters in Table 1, where the total of cases is 1675 and other experiment

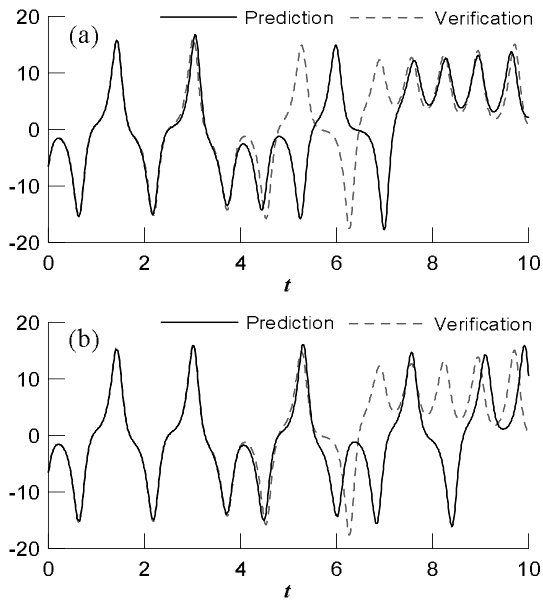

Fig. 1. Predicted and verified x curves under the condition of $\delta \boldsymbol{r}$=0.1 based on the (a) PDM and (b) ADM respectively.

Table 1. Comparison between global mean VT based on the PDM and ADM respectively in terms of different disturbance schemes of model parameters.

Model error vectors	Disturbance schemes of model parameters	Global mean PDM-VT	Global mean ADM-VT	Case number: ADM-VT> PDM-VT	Percentage: ADM-VT> PDM-VT
$(0, \delta\boldsymbol{r}x, 0)^{\mathrm{T}}$	$\delta\boldsymbol{r} = 1.0$	2.38	5.03	1625	97.0%
$(0, \delta\boldsymbol{r}x, 0)^{\mathrm{T}}$	$\delta\boldsymbol{r} = 0.1$	4.92	6.42	1345	80.3%
$(0, \delta\boldsymbol{r}x, 0)^{\mathrm{T}}$	$\delta\boldsymbol{r} = 0.01$	7.40	6.95	612	36.5%
$(-\delta\sigma x + \delta\sigma y, 0, 0)^{\mathrm{T}}$	$\delta\sigma = 1.0$	4.46	4.64	970	57.9%
$(-\delta\sigma x + \delta\sigma y, 0, 0)^{\mathrm{T}}$	$\delta\sigma = 0.1$	6.60	7.41	1065	63.6%
$(-\delta\sigma x + \delta\sigma y, 0, 0)^{\mathrm{T}}$	$\delta\sigma = 0.01$	9.00	7.13	236	14.1%
$(0, 0, -\delta bz)^{\mathrm{T}}$	$\delta b = 1.0$	1.38	5.05	1671	99.8%
$(0, 0, -\delta bz)^{\mathrm{T}}$	$\delta b = 0.1$	3.35	6.13	1594	95.2%
$(0, 0, -\delta bz)^{\mathrm{T}}$	$\delta b = 0.01$	5.92	6.89	1144	68.3%
$(\delta\sigma(y - x), \delta\boldsymbol{r}x, -\delta bz)^{\mathrm{T}}$	$\delta\boldsymbol{r}, \delta\sigma, \delta b = 0.1$	3.50	6.08	1578	94.2%

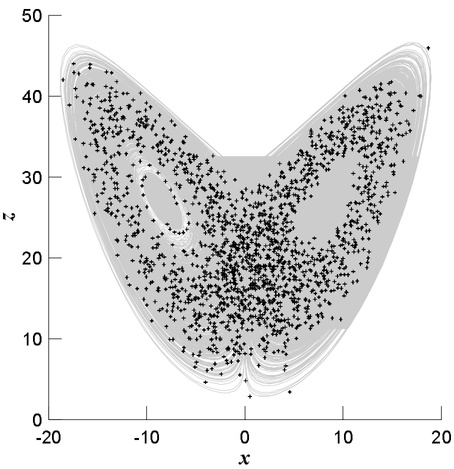

Fig. 2. Distributions of 1675 initial values in x-z phase plane.

designs are the same as those above in case study. Here, the initial values used in global experiments are obtained by taking one in every 20 time steps from the latest 5-year simulated observe data (refer to little plus signs in Fig. 2). These initial values with global sense distribute nearly homogenously in phase space. But relatively speaking, there exist fewer cases near two stable equilibriums and outer boundary of chaotic attractor. Moreover, because the period of prediction based on every initial value is generally 2000–3000 time steps, there are 1675 groups of initial values in total in the global experiments.

Firstly, Table 1 presents the comparison between global mean VT based on the PDM and ADM respectively in terms of different disturbance schemes of model parameters.

It can be clearly seen from Table 1 that the ADM has much higher performance than the PDM in most cases. Especially, when all of three parameters are equal to 0.1, the cases that global mean VT corresponding to the ADM is longer than that to the PDM, and have a percentage of 94.2% in all cases. These results clearly show that introducing the information of historical analogue in the ADM helps to prolong the valid time of dynamical prediction further which can be more clearly and visually documented in Fig. 3. By examining distributions of the VT corresponding to 1675 initial values in x-z phase plane under the condition of $\delta\boldsymbol{r} = 0.1$ based on the PDM and ADM respectively, we find out that the PDM with parameter error is good at prediction near two stable equilibriums and on the bottom of attractor, whereas the ADM has more homogeneous distributions of the VT. Over most areas in phase plane, the VT corresponding to the ADM is larger than that to the PDM.

Also, we note that every parameter error has different impacts on prediction results, which is evidently related to the proportion of given parameter disturbances in themselves. Generally speaking, the larger the model error is, the less the predictive effect of the dynamical model is. For example, according to Table 1, the global mean VT corresponding to different parameters based on the PDM gets larger with the lessening of parameter errors, which is also documented by the results of most disturbance schemes in the ADM except $\delta\sigma = 0.01$. Besides, when $\delta\boldsymbol{r}$ or $\delta\sigma$ is equal to 0.01, the global mean VT corresponding to the ADM is smaller than that to the PDM and the percentage of the former in all cases is lower than 50%, which shows that at this time the ADM is unsatisfactory.

The above analyses show that when model errors represented by parameter disturbances are relatively large, the ADM can quite effectively reduce prediction errors and prolong the valid time of prediction. Due to different predictability for different initial values in phase space (Chou, 1995), there still exist a few cases in which the PDM has longer valid time than the ADM. On the other hand, when such model errors are

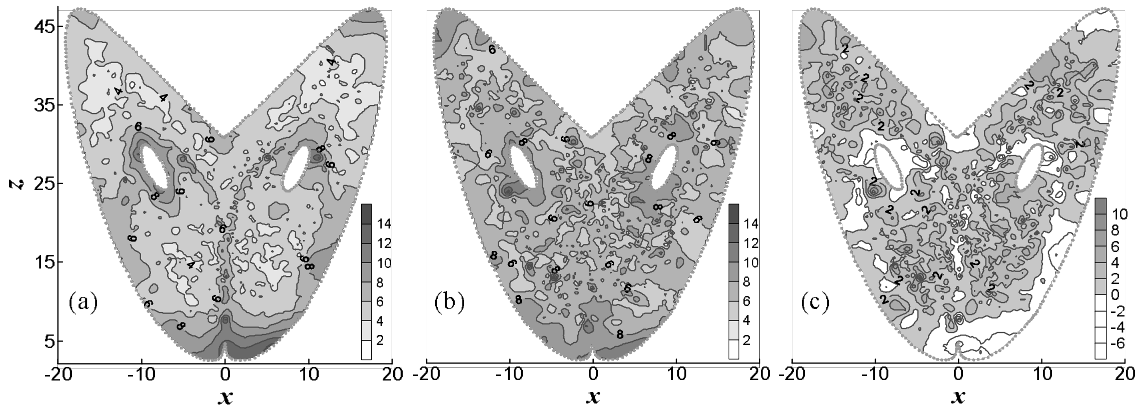

Fig. 3. Distributions of the VT corresponding to 1675 initial values in x-z phase plane under the condition of $\delta r = 0.1$ based on the (a) PDM, (b) ADM and (c) ADM-PDM respectively.

considerably small, the predictive effect of the PDM is significantly improved, whereas comparatively, that of the ADM is slowly promoted and even not as good as the former, which may be seen in the VT distributions (figure omitted).

7. Global experiments with multi-reference-state updating

As above, we can assume that if a real numerical model is perfectly developed in the future, the ADM will seem to have less and less advantage compared with the PDM. However, the above results only are based on the limited historical analogues with given quantity and frequency, where there likely exists the problem of lacking analogy information in the ADM. Since model errors inevitably exist in real numerical models, it will be reasonably believed that the ADM can have better performance than the PDM by introducing more analogy information, even though model errors are rather small, which will be further discussed in the following work.

7.1 *The method based on multi-reference-state updating*

As we know, there are usually many analogues similar to the current initial value in history. The ensemble of many historical analogues is often used in traditional statistical analogue prediction (SAP) (Barnett and Preisendorfer, 1978; van den Dool, 1987; Toth, 1989; Livezey et al., 1994), which presents important reference for the dynamical analogue prediction (DAP) that is intended to effectively utilize historical analogy information in dynamical prediction and to realize the adequate combination of dynamical and statistical methods. A new method named multi-reference-state updating (MRSU) has been developed to comprehensively consider the multi-analogue information and

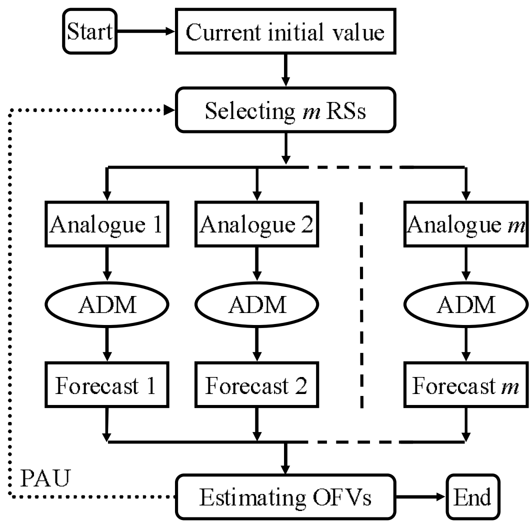

Fig. 4. Schematic illustration of the MRSU.

updating of analogue because the similarity between the current initial value and historical analogue can only persist for a very limited time in the process of prediction (Ren and Chou, 2006b; Ren et al., 2006).

It can be clearly seen from Fig. 4 that in the MRSU, according to the idea of "updating", multi-reference states are newly selected on the period of analogue updating (PAU) in integrating the ADM and optimal forecast vectors (OFVs) are estimated from multi-forecasts generated by the ADM based on certain methods. Such the "selection-estimation" cycle is repeatedly operated until the whole forecast is completed. The methodology associated with estimating the OFV is the same as that used for estimating new prediction errors from historical analogical prediction errors in the final ACE method, which has been introduced in detail in the literature (Ren and Chou, 2007b). Here, it needs be noted that the Hyperplane approximation method (HAM), as a theoretical me-

Table 2. Comparison between global mean VT based on the PDM and MRSU respectively in terms of different disturbance schemes of model parameters.

Model error vectors	Disturbance schemes of model parameters	Global mean PDM-VT	Global mean MRSU-VT	Case number: MRSU-VT> PDM-VT	Percentage: MRSU-VT> PDM-VT
$(0, \delta \boldsymbol{r} x, 0)^{\mathrm{T}}$	$\delta \boldsymbol{r} = 1.0$	2.38	7.03	1675	100%
$(0, \delta \boldsymbol{r} x, 0)^{\mathrm{T}}$	$\delta \boldsymbol{r} = 0.1$	4.92	9.17	1653	98.7%
$(0, \delta \boldsymbol{r} x, 0)^{\mathrm{T}}$	$\delta \boldsymbol{r} = 0.01$	7.40	9.55	1526	91.1%
$(-\delta\sigma x + \delta\sigma y, 0, 0)^{\mathrm{T}}$	$\delta\sigma = 0.01$	9.00	9.54	1026	61.3%
$(\delta\sigma(y-x), \delta \boldsymbol{r} x, -\delta b z)^{\mathrm{T}}$	$\delta \boldsymbol{r}, \delta\sigma, \delta b = 0.1$	3.50	9.51	1675	100%

thod, sometimes appears unsuccessful in solving linear algebraic equation groups due to uneven phase curve-plane. At this time, the HAM will be replaced by a simple linear estimation method (SLEM) (also refer to Ren and Chou, 2007b), which will be proved to be reasonable in the following experiments.

In previous work, although experiments for monthly DERF by a simplified MSRU displayed some exciting preliminary results, it will be very difficult for real atmospheric ADMs to conduct comprehensive experiments under the conditions of various predictabilities. Thus, to examine the predictive effect of the MRSU for the above difficulty that is induced by considerably small model errors, as well as to discuss other important issues, e.g., the impacts of model errors and key parameters on prediction of the MRSU, the global and sensitive experiments will be further conducted in the following.

7.2 *Global experiments based on the MRSU*

Here, the experiments based on the PDM are still regarded as the contrast of those based on the MRSU. First, the performances of the MRSU are examined in terms of different disturbance schemes of model parameters, which need to pre-assign some key parameters. Here, without losing generality, take parameters PAU=20 steps and IOS=5 steps. The SLEM based on the first 4 best analogues is employed to estimate the OFVs. Considering that the differences between the verified results corresponding to individual disturbance schemes of three model parameters in Table 1 are quite small, Table 2 gives some representative results.

Compared with Table 1, it can be more clearly seen from Table 2 that the MRSU on the first 4 best analogues has far higher performance not only than the PDM but also the ADM only based on the single best analogue. This is true especially when either $\delta \boldsymbol{r}$ or all of three parameters are equal to 0.1, the cases that global mean VT corresponding to the MRSU is longer than those to the PDM, almost reach a percentage of 100% in all cases. Moreover, the two unsatisfactory situations corresponding to the ADM, when $\delta \boldsymbol{r}$ or $\delta\sigma$ is equal to 0.01, have also been successfully overcome. All the results show that the MRSU superimposing on the ADM has the capability for solving the difficulty from small model parameter errors by introducing and updating multi- historical analogues.

7.3 *Sensitive experiments based on the MRSU*

In the above experiments, the MRSU displays its validity by using 4 analogues. Then, how will the change of the number of selected analogues have the impacts on predictive effectiveness based on the MRSU? In contrast to the verified results of the PDM and HAM, Fig. 5 presents the results of the MRSU based on the HAM and SLEM with different number of selected analogues. For the convenience of comparison, the global mean VT (=4.92) of the PDM is regarded as a reference.

Evidently in Fig. 5, the VT of the SLEM and HAM is far longer than that of the PDM because the information of analogue is utilized in the formers, where the HAM exhibits the best performance by accurately solving the linear algebraic equation group satisfied by the OFVs. The verified effects of the SLEM become better with the increase of selected analogues and gradually tend to that of the HAM. Indeed, the VT of the SLEM cannot increase by degrees linearly and decreases after about 40 analogues. Once the total quantity of historical observed samples is prescribed, the quality of analogues introduced into the MRSU will gradually decrease and the analogues with low similarity may have a negative accumulative contribution to the predictive effect. In other words, there likely exists an optimal number of selected analogues in the MRSU. Besides, the SLEM has slightly lower verified scores than the HAM by introducing adequate multi-analogues and can be regarded as the suitable simplified version of the HAM in practical applications.

Furthermore, the similarity between the two states cannot persist for a very long time, so it is very necessary to update the analogue repeatedly in the process of integrating the ADM. The PAU should reflect the

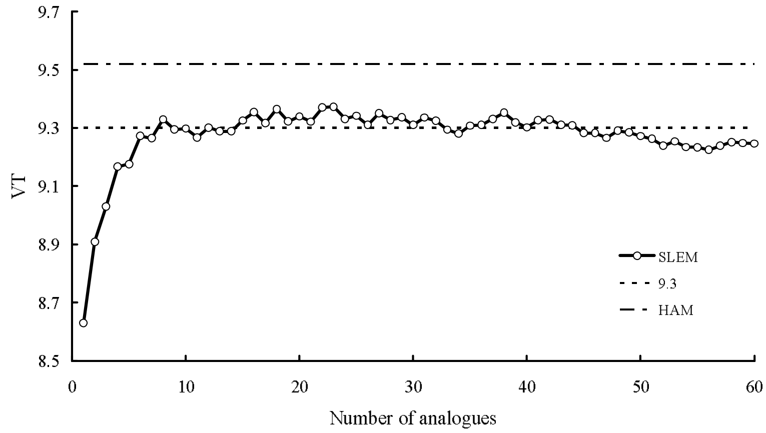

Fig. 5. Global mean VT as a function of the number of selected analogues based on the MRSU in terms of the SLEM and HAM respectively, where the disturbance schemes of $\delta \boldsymbol{r} = 0.1$.

Table 3. Global mean VT based on the MRSU in terms of different PAU and IOS (Unit: time step number) respectively by the disturbance schemes of $\delta \boldsymbol{r} = 0.1$.

PAU	IOS				
	1	5	10	20	40
1	14.02	-	-	-	-
5	13.94	9.70	-	-	-
10	14.20	9.84	7.31	-	-
20	14.47	9.52	6.95	5.05	-
40	13.60	9.38	6.82	4.21	1.77

global mean persistence between any two states and may be approximately determined by trial and error. Moreover, as has been known, the quantity of available historical samples that may be represented indirectly as the IOS by selecting samples in terms of different intervals from total observed dataset is vital for the methodology based on analogy information. Thus, the impacts of different PAU and IOS on the MRSU prediction will be deep examined. Here, the HAM based on 4 analogues is employed for estimating the OFVs to acquire the theoretical upper limit of prediction skill of the MRSU.

It can be easily seen from Table 3 that the global mean VT significantly increase with the decrease of the IOS. Comparatively, the PAU seems to have an optimal extremum for certain IOS and the predictive effect based on the MRSU will be influenced on the whole when the PAU is bigger or smaller than the optimal extrema. This could be corresponding to the global mean persistence between any two states. Moreover, the IOS that represents the density and quantity of historical data plays a very important role in the MRSU, and its impacts on the predictive effect even are more significant than those of the PAU.

The above experiment results show that the impacts of model errors in numerical models on prediction may be overcome to some extent by introducing the analogy information in a historical dataset. The MRSU can effectively reduce prediction errors by updating analogues and using multi-reference states as well as estimating the OFVs. Furthermore, for certain IOS, there exists the optimum of the PAU, and under the condition of giving suitable PAU, the valid time of prediction based on the MRSU can be evidently prolonged with the increase of observed samples. Also, the number of selected analogues has significant impacts on the MRSU prediction in terms of the SLEM that may be regarded as the suitable replacement of the HAM in practical application.

8. Summary

In previous studies, the essential idea of analogue-dynamical model (ADM) has been proposed and related numerical prediction experiments on quasi-geostrophic models have been conducted. Exciting initial results showed that the ADM can effectively combine statistical and dynamical methods together, which promote us to examine the predictability and development problems associated with the ADM. Thus in the current work, we employ the Lorenz system and study the theoretical basis and application of the ADM. We divide model errors into three parts such as systematic error, time-dependent error and stochastic error. Primary theoretical analyses show that the ADM can significantly reduce systematic model errors when a time-independent model error vector is introduced into the Lorenz model. Further, by appending small perturbations on original model parameters, we theoretically examine the impacts of time-dependent model errors and results show that the ADM has less model errors than the pure dynamical model and can effectively utilize atmospheric analogy information in traditional dynamical prediction.

Case studies and global experiments with a lot of initial values scattering over phase space are conducted in order to examine the characteristics of predictability of the ADM in the Lorenz system. The results show that the ADM can quite effectively reduce prediction errors and prolong the valid time of prediction in most situations and have better performance than the PDM, but when model errors is considerably small, the latter will be superior to the former. To overcome such a problem by using more analogy information, the method based on multi-reference-state updating (MRSU) that can introduce the information of multi-analogue and update analogue is further employed in the process of integrating the ADM. Experiments exhibit exciting results based on the MRSU, and some key factors such as the PAU, the quantity of observed samples and the number of selected analogues, have different impacts on prediction of the MRSU. These conclusions present valuable references for practical prediction. Indeed, how to a priori determine the number of analogues used in the SLEM and the optimal PAU for certain quantity of historical samples, needs to be further studied in future works.

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