SOME PROPERTIES OF THE SOLUTIONS OF THE LARGE-SCALE EQUATIONS OF ATMOSPHERE*

——NONLINEAR ADJUSTMENT PROCESS BY EXTERIOR STEADY SOURCES

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ABSTRACT

From the large-scale equations of atmospheric motion, we investigated the long-time behaviour of atmospheric system forcing by exterior steady sources. Firstly, we established the fundamental functional space and operator equations, and then demonstrated the existence and uniqueness theorems of solutions. Based on these results, the existence of the bounded global absorbing set and invariant set in it were discussed. Finally, the nonlinear adjustment process to exterior sources was revealed.

Keywords: solutions of atmosphere equations, nonlinear adjustment process, absorbing set.

I. INTRODUCTION

Long-range numerical forecasts and climatic theories deal with the long-time behaviour of the atmospheric system. Thus, before we design the model, its characteristics should be understood. To lay a more solid mathematical and physical foundation, the study of basic theories needs to be made, and a more strict theory and new computing method in this respect were established.

We started by studying the overall asymptotic behaviour of the atmospheric system and its dependence on the surrounding conditions under the ideal assumption (i.e. steady situation or strict period). Chou[1,2] first discussed the nonlinear adjustment process of the atmospheric system adjusted to exterior sources in n-dimensional space. He found that there exists a bounded global absorbing set in R^n space and no matter how initial conditions are, the state of system will develop to the bounded absorbing set with the increase of time. We guess that the conclusions

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are also tenable in the infinite dimensional Hilbert space. The purpose of this paper is to verify it and do further investigation.

Firstly, we established the fundamental functional space and operator equations, and then demonstrated the existence and uniqueness theorems of solutions. Based on this, the existence of the bounded global absorbing set and invariant set in it were discussed. Finally, the physical meaning of solutions was explained.

II. The Model

As we are interested in the large-scale motion of the atmosphere, we use the following equations in spherical coordinates[2]:

$$\frac{\partial V_\lambda}{\partial t} + \mathscr{L} V_\lambda + \left(2\Omega\cos\theta + \frac{\mathrm{ctg}\theta}{a} V_\lambda\right) V_\theta + \frac{1}{a\sin\theta}\frac{\partial \phi}{\partial \lambda} - \frac{\partial}{\partial p}\left[\nu_1\left(\frac{gp}{R\overline{T}}\right)^2 \frac{\partial V_\lambda}{\partial p}\right] - \mu_1\nabla^2 V_\lambda = 0, \tag{2.1}$$

$$\frac{\partial V_\theta}{\partial t} + \mathscr{L} V_\theta - \left(2\Omega\cos\theta + \frac{\mathrm{ctg}\theta}{a} V_\lambda\right) V_\lambda + \frac{1}{a}\frac{\partial \phi}{\partial \theta} - \frac{\partial}{\partial p}\left[\nu_1\left(\frac{gp}{R\overline{T}}\right)^2 \frac{\partial V_\theta}{\partial p}\right] - \mu_1\nabla^2 V_\theta = 0, \tag{2.2}$$

$$\frac{\partial \phi}{\partial p} + \frac{R}{p} T = 0, \tag{2.3}$$

$$\frac{R^2}{C^2}\frac{\partial T}{\partial t} + \frac{R^2}{C^2}\mathscr{L} T - \frac{R}{p}\omega - \frac{\partial}{\partial p}\left[\nu_2\left(\frac{gp}{R\overline{T}}\right)^2 \frac{\partial T}{\partial p}\right] - \mu_2\nabla^2 T = \frac{R^2}{C^2}\frac{\varepsilon}{C_p}, \tag{2.4}$$

$$\frac{1}{a\sin\theta}\left[\frac{\partial V_\lambda}{\partial \lambda} + \frac{\partial V_\theta \sin\theta}{\partial \theta}\right] + \frac{\partial \omega}{\partial p} = 0, \tag{2.5}$$

where

$$\mathscr{L} = \frac{V_\lambda}{a\sin\theta}\frac{\partial}{\partial\lambda} + \frac{V_\theta}{a}\frac{\partial}{\partial\theta} + \omega\frac{\partial}{\partial p},$$

$$\nabla^2 = \frac{1}{a^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{a^2\sin\theta}\frac{\partial^2}{\partial\lambda^2},$$

$$C^2 = \frac{R^2\overline{T}}{g}(\gamma_d - \gamma).$$

$\overline{T} = \overline{T}(P)$ is the time mean on the isobaric p surface, T the deviation from the $\overline{T}$, ϕ the deviation from $\bar{\phi}$, ε the diabatic heating rate of the atmosphere, other symbols are always used in meteorology.

We discuss the motion around the earth globally. So the domain here is $\Omega = S^2 \times (p_0, P)$ with $p_0 > 0$, and thus the boundary value conditions are

$$p = P,\ \boldsymbol{V} = 0,\ \omega = 0,\ \frac{\partial T}{\partial p} = \alpha_s(T_s - T) \tag{2.6}$$

and $$p=p_0,\ \frac{\partial V}{\partial p}=0,\ \omega=0,\ \frac{\partial T}{\partial p}=0, \tag{2.7}$$

where T is the temperature on the surface of earth, α_s the parameter relevant to the turbulent heat conduction rate and dependent on the characteristics of the surface.

III. Basic Space and Operator Equation

Let W_0 be the closure of $C_0^\infty(\Omega)$ with the following norm:

$$\|\omega\| = \left(\int_\Omega\left(\omega^2 + \left(\frac{\partial\omega}{\partial p}\right)^2\right) dS^2 dp\right)^{1/2}.$$

Then it is easy to see that W_0 is a Hilbert space (see Ref.[4]) and can be endowed with the following equivalent norm:

$$\|\omega\| = \left(\int_\Omega\left(\frac{\partial\omega}{\partial p}\right)^2 dS^2 dp\right)^{1/2}. \tag{3.1}$$

Let $T\Omega|_{S^2}$ be the restriction of the tangent bundle of Ω on S^2. Then a smooth section of $T\Omega|_{S^2}$ is a smooth vector field on Ω with values in the tangent space of S^2.

Let $C^\infty_{P,0}(T\Omega|_{S^2})$ denote those sections of $T\Omega|_{S^2}$, taking value zero near $S^2\times\{P\}$. And then we can define

$$\mathscr{V} = \left\{(\boldsymbol{V},\omega)\in C^\infty_{P,0}(T\Omega|_{S^2})\times C_0^\infty(\Omega)\,|\,\nabla\cdot\boldsymbol{V}+\frac{\partial\omega}{\partial p}=0\right\},$$

$$\mathscr{V}_T = \mathscr{V}\times C^\infty(\Omega);$$

and

$$\boldsymbol{V}_T = \text{the closure of } \mathscr{V}_T \text{ in } H^1(T\Omega|_{S^2})\times W_0\times H^1(\Omega),$$
$$\boldsymbol{H}_T = \text{the closure of } \mathscr{V}_T \text{ in } L^2(T\Omega)\times L^2(\Omega),$$

here $H^1(\Omega)$ is the standard Sobolev space, $H^1(T\Omega|_{S^2})$ the Hilbert space made by those sections of $T\Omega|_{S^2}$ with all the first order derivatives belonging to L^2 (see Ref. [5]).

Using the diagnosis equation (2.5), there are obviously two constants m_1, $m_2>0$ such that

$m_1(\|\boldsymbol{V}\|^2+\|T\|^2)\leqslant\|\boldsymbol{V}\|^2+\|\omega\|^2+\|T\|^2\leqslant m_2(\|\boldsymbol{V}\|^2+\|T\|^2)$, where $\|\boldsymbol{V}\|$ takes $H^1(T\Omega|_{S^2})$-norm, $\|T\|$ the $H^1(\Omega)$-norm and $\|\omega\|$ the W_0-norm.

So we can use the following equivalent norm in $\boldsymbol{V}_T$ (see Ref. [4]):

$$\|\varphi\| = (\|\boldsymbol{V}\|^2+\|T\|^2)^{1/2},\ \forall\varphi=(\boldsymbol{V},\omega,T)\in\boldsymbol{V}_T.$$

In order to handle the second order terms, now we define a linear operator A: $\boldsymbol{V}_T\to\boldsymbol{V}_T^*$ by

$$\begin{aligned}\langle A\varphi,\varphi_1\rangle = \int_\Omega\Big[&\mu_1\nabla\boldsymbol{V}\cdot\nabla\boldsymbol{V}_1+\nu_1\left(\frac{gp}{R\overline{T}}\right)^2\frac{\partial\boldsymbol{V}}{\partial p}\cdot\frac{\partial\boldsymbol{V}_1}{\partial p}\\ &+\mu_2\nabla T\cdot\nabla T_1+\nu_2\left(\frac{gp}{R\overline{T}}\right)^2\frac{\partial T}{\partial p}\cdot\frac{\partial T_1}{\partial p}\Big]dS^2dp\\ &+\int_{S^2\times\{P\}}\nu_2\left(\frac{gp}{R\overline{T}}\right)^2\alpha_s TT_1 dS^2,\end{aligned} \tag{3.2}$$

where $\varphi = (\boldsymbol{V}, \omega, T)$, $\varphi_1 = (\boldsymbol{V}_1, \omega_1, T_1) \in \boldsymbol{V}_T$ and $\boldsymbol{V}_T^*$ is the dual space of $\boldsymbol{V}_T$.

Then it is obvious (see Ref. [4]) that there exist positive constants C_1, C_2 such that

$$C_1\|\varphi\|^2 \leqslant \langle A\varphi, \varphi\rangle \leqslant C_2\|\varphi\|^2, \quad \forall \varphi \in \boldsymbol{V}_T. \tag{3.3}$$

As for other parts appearing in the equation, we let

$$\boldsymbol{V}_k = \boldsymbol{V}_T \cap (H^k(T\Omega) \times H^k(\Omega)), \quad (k \geqslant 0),$$

then we can define a trilinear functional b_1: $\boldsymbol{V}_T \times \boldsymbol{V}_T \times \boldsymbol{V}_3 \to R$ by

$$b_1(\varphi, \varphi_1, \varphi_2) = \int_\Omega \Big[\Big(\boldsymbol{V} \cdot \nabla \boldsymbol{V}_1 + \omega \frac{\partial \boldsymbol{V}_1}{\partial p}\Big) \cdot \boldsymbol{V}_2 + \frac{R^2}{C^2}\Big(\boldsymbol{V} \cdot \nabla T_1 + \omega \frac{\partial T_1}{\partial p}\Big) \cdot T_2 - \Big(\Big(\boldsymbol{k} \cdot \Big(\frac{\boldsymbol{V} \cdot \nabla \sin\theta}{a\sqrt{G}}\Big)\Big) \boldsymbol{k} \times \boldsymbol{V}_1\Big) \cdot \boldsymbol{V}_2\Big] dS^2 dp, \tag{3.4}$$

where $\varphi = (\boldsymbol{V}, \omega, T)$, $\varphi_1 = (\boldsymbol{V}_1, \omega_1, T_1) \in \boldsymbol{V}_T$, $\varphi_2 = (\boldsymbol{V}_2, \omega_2, T_2) \in \boldsymbol{V}_3$, G is the determinant of the Riemannian metric matrix, and θ is the colatitude. In spherical coordinates, $G = \sin\theta$ and b_1 represents the advective and curature terms.

Moreover, we define a bilinear functional b_2: $\boldsymbol{V}_T \times \boldsymbol{V}_T \to R$ by

$$b_2(\varphi, \varphi) = \int_\Omega \Big[\frac{R}{p}(T_1\omega_2 - T_2\omega_1) + 2\Omega\cos\theta(\boldsymbol{k} \times \boldsymbol{V}) \cdot \boldsymbol{V}_2\Big] dS^2 dp,$$

$$\forall \varphi_i = (\boldsymbol{V}_i, \omega_i, T_i) \in \boldsymbol{V}_T, \quad (i = 1, 2),$$

then, obviously, by the diagnosis equation (2.5) and integrating by parts, b_1 and b_2 are well-defined and satisfy

$$b_1(\varphi, \varphi_1, \varphi_1) = 0, \quad \forall \varphi \in \boldsymbol{V}_T, \ \varphi_1 \in \boldsymbol{V}_3, \tag{3.5}$$

$$b_2(\varphi_1, \varphi_1) = 0, \quad \forall \varphi_1 \in \boldsymbol{V}_T. \tag{3.6}$$

The above two equalities embody the physics fact that the energy is invariant under the actions reflected by b_1 and b_2.

Proposition 3.1 (*see Ref.* [4]). *Assume that*

$$\alpha_s \in C^2(S^2), \quad T_s \in H^2(S^2), \quad 0 < m \leqslant \alpha_s \leqslant M < \infty, \tag{3.7}$$

then there is $T^* \in H^2(\Omega)$ *such that*

$$|b_1(\varphi, \psi, \varphi)| \leqslant \frac{1}{2} C_1\|\varphi\|^2, \quad \forall \varphi \in \boldsymbol{V}_T \tag{3.8}$$

and

$$\frac{\partial T^*}{\partial p}\Big|_{\partial\Omega} = \begin{cases} \alpha_s(T_s - T^*), & p = P, \\ 0, & p = p_0, \end{cases}$$

where $\psi = (0, 0, 0, T^*)$, *and* C_1 *is given by* (3.3).

With the above preparation, we can easily see that the initial-boundary value problems (2.1—2.7) are equivalent to the following Cauchy problem:

Problem 3.1. Find $\varphi \in L^2(0, \tau; \boldsymbol{V}_T)(\tau > 0)$ such that

$$B\varphi \in L^\infty(0, \tau; L^2(T\Omega) \times L^2(\Omega)),$$

$$\frac{\partial}{\partial t}B\varphi + A\varphi + N_1(\varphi,\varphi) + N_2(\varphi) + N_1(\varphi,\psi) + N_1(\psi,\varphi) = \boldsymbol{f}, \tag{3.9}$$

$$B\varphi|_{t=0} = B\varphi_0,$$

where (3.9) makes sense in $\boldsymbol{V}_3^*$, $\varphi_0 \in \boldsymbol{H}_T$ and B is the diagonal matrix $B = \mathrm{diag}(1, 1, 0, R^2/C^2)$. For any φ, $\varphi_1 \in \boldsymbol{V}_T$, the operators $N_1(\varphi, \varphi_1) \in \boldsymbol{V}_3^*$ and $N_2(\varphi) \in \boldsymbol{V}_T^*$ are defined by

$$\langle N_1(\varphi, \varphi_1), \varphi_2\rangle = b_1(\varphi, \varphi_1, \varphi_2), \ \forall \varphi_2 \in \boldsymbol{V}_3,$$
$$\langle N_2(\varphi_1), \varphi_2\rangle = b_2(\varphi_1, \varphi_2), \ \forall \varphi_2 \in \boldsymbol{V}_T.$$

Moreover, in (3.9), one has

$$\boldsymbol{f} = \left(0,\ 0,\ -\frac{R}{p}T^*,\ \frac{R^2}{C^2}\frac{\varepsilon}{C_p} + \mu_2\nabla^2 T^* + \nu_2\frac{\partial}{\partial p}\left(\left(\frac{gp}{R\overline{T}}\right)^2\frac{\partial T^*}{\partial p}\right)\right). \tag{3.10}$$

The equivalence of Problem 3.1 and the original problem means that if φ is a solution of Problem 3.1, then $\varphi + \psi$ is the one of the original problem. Conversely, if $\tilde{\varphi}$ is a solution of the original one, then $\tilde{\varphi} - \psi$ is a solution of Problem 3.1.

Obviously, (3.9) in Problem 3.1 is also equivalent to the following variational equation:

$$\frac{d}{dt}(B\varphi,\varphi_1) + \langle A\varphi,\varphi_1\rangle + b_1(\varphi,\varphi,\varphi_1) + b_2(\varphi,\varphi_1) + b_1(\varphi,\psi,\varphi_1)$$
$$+ b_1(\psi,\varphi,\varphi_1) = \langle \boldsymbol{f},\varphi_1\rangle, \ \forall \varphi_1 \in \boldsymbol{V}_3. \tag{3.11}$$

Then we have the following existence theorem which is proved in Ref. [4] by the method of finite differences with respect to the time t.

Theorem 3.1. *Under the assumption* (3.7), *if* $\varepsilon \in L^2(0,\tau;(H^1(\Omega))^*)$, *then there is at least one solution of Problem* 3.1 *such that*

$$\varphi \in L^2(0,\tau;\boldsymbol{V}_T), \ B\varphi \in L^\infty(0,\tau;L^2(T\Omega)\times L^2(\Omega)),$$
$$|B_1\varphi|^2 + \frac{1}{2}C_1\int_0^t \|\varphi(t)\|^2 dt \leqslant |B_1\varphi_0|^2 + \frac{1}{C_1}\int_0^t \|\boldsymbol{f}(t)\|^2_{V_T^*}dt,\ t\in[0,\tau] \text{ a.e.}, \tag{3.12}$$

where $B_1 = \mathrm{diag}(1, 1, 0, R/C)$ *and* $|B_1\varphi|$ *takes* L^2*-norm.*

Remark 1. (3.12) is better than the corresponding inequality in Ref. [4], but the proofs are the same.

Remark 2. It follows from (3.12) that there is a constant $C_1 > 0$ such that

$$|B_1\varphi|^2 + \widetilde{C}_1\int_0^t |B_1\varphi|^2 dt \leqslant |B_1\varphi_0|^2 + \frac{1}{C_1}\int_0^t \|\boldsymbol{f}(t)\|^2_{V_T^*}dt.$$

So one has

$$|B_1\varphi|^2 \leqslant \left\{|B_1\varphi_0|^2 + \frac{1}{C_1}\int_0^t e^{\tilde{C}_1 t}\|\boldsymbol{f}(t)\|^2_{V_T^*}dt\right\}e^{-\tilde{C}_1 t}, \text{ a.e., } t\in[0,\tau]. \tag{3.13}$$

IV. A Uniqueness Theorem

Theorem 3.1 gives an existence theorem for weak solutions. However, we do not know any uniqueness of the solutions. So in this section, we show a uniqueness theorem under some regularity assumption for the solutions.

Theorem 4.1. *If* $T^* \in H^3(\Omega)$, *then there is at most one solution of Problem* 3.1 *such that*

$$B\varphi \in L^2(0,\tau;H^3(T\Omega) \times H^3(\Omega)), \tag{4.1}$$

$$\varphi \in L^2(0,\tau;H^2(T\Omega) \times H^2(\Omega)). \tag{4.2}$$

Proof. Since

$$|b_1(\varphi, \varphi, \varphi_1)| \leqslant C\|\varphi\|_{H^2} \cdot |B\varphi|_{L^2} \cdot \|\varphi_1\|,$$

one has $\|N_1(\varphi,\varphi)\|_{V_T^*} \leqslant C\|\varphi\|_{H^2} \cdot |B\varphi|_{L^2}$.

By (4.2) and $B\varphi \in L^\infty(0,\tau;L^2(T\Omega) \times L^2(\Omega))$, one obtains

$$N_1(\varphi, \varphi) \in L^2(0, \tau; \boldsymbol{V}_T^*).$$

Similarly, one can prove that $N_2(\varphi), N_1(\psi, \varphi), N_1(\varphi, \psi) \in L^2(0, \tau; \boldsymbol{V}_T^*)$.

So Eq. (3.9) implies that

$$B\varphi' \in L^2(0, \tau; \boldsymbol{V}_T^*), \tag{4.3}$$

and $B\varphi$ is a continuous function from $[0,\tau]$ into $L^2(T\Omega) \times L^2(\Omega)$ after possibly a modification on a set of measure zero.

Now suppose φ_1 and φ_2 are two solutions of Problem 3.1 satisfying (4.1—4.2) and let $\varphi = \varphi_1 - \varphi_2$. Then it is easy to see that

$$\frac{d}{dt}|B_1\varphi|^2 + 2\langle A\varphi,\varphi\rangle + 2b_1(\varphi_1,\varphi_1,\varphi) - 2b_1(\varphi_2, \varphi_2, \varphi) + 2b_2(\varphi, \varphi)$$
$$+ 2b_1(\varphi,\psi,\varphi) + 2b_1(\psi,\varphi,\varphi) = 0, \tag{4.4}$$

i.e.

$$\frac{d}{dt}|B_1\varphi|^2 + 2C_1\|\varphi\|^2 \leqslant 2|b_1(\varphi,\varphi_2,\varphi)| + |b_1(\varphi,\psi,\varphi)|. \tag{4.5}$$

But $|b_1(\varphi, \varphi_2,\varphi)| \leqslant C|\varphi| \cdot |B_1\varphi| \cdot \|B_1\varphi_2\|_{H^3} \leqslant \frac{C_1}{4}\|\varphi\|^2 + \widetilde{C}|B_1\varphi|^2 \cdot \|B\varphi_2\|_{H^3}$, (4.6)

and
$$|b_1(\varphi, \psi, \varphi)| \leqslant \frac{C_1}{4}\|\varphi\|^2 + \widetilde{C}|B_1\varphi|^2 \cdot \|\psi\|_{H^3}. \tag{4.7}$$

Consequently,

$$\frac{d}{dt}|B_1\varphi|^2 + C_1\|\varphi\|^2 \leqslant \widetilde{C}(\|\psi\|_{H^3} + \|B_1\varphi_2\|_{H^3}) \cdot |B_1\varphi|^2,$$

i.e.

$$\frac{d}{dt}|B_1\varphi|^2 \leqslant \widetilde{C}(\|\psi\|_{H^3} + \|B_1\varphi_2\|_{H^3}) \cdot |B_1\varphi|^2.$$

Therefore, $|B_1\varphi|^2 \leqslant 0$ and then $\varphi_1 \equiv \varphi_2$, hence the result.

V. A Global Absorbing Set

For simplicity, let ε be time-independent. As for the case of ε depending on t, we can discuss similarly. By Theorem 3.1, we can see that Problem 3.1 has solution φ such that

$$\varphi \in L^2_{loc}(0,\infty; \boldsymbol{V}_T),\ B\varphi \in L^\infty_{loc}(0,\infty; L^2(T\Omega)\times L^2(\Omega)), \tag{5.1}$$

and the energy inequality (3.12).

Since ε is time-independent, by (3.12) one has

$$|B_1\varphi(t)|^2 + \frac{1}{2}C_1\int_0^t \|\varphi(t)\|^2 dt \leqslant |B_1\varphi_0|^2 + \frac{t}{C_1}\|\boldsymbol{f}\|^2_{V_T^*}.$$

It follows that

$$|B_1\varphi(t)|^2 + \widetilde{C}_1\int_0^t |B_1\varphi(t)|^2 dt \leqslant |B_1\varphi_0|^2 + \frac{t}{C_1}\|\boldsymbol{f}\|^2_{V_T^*},$$

for some constant $\widetilde{C}_1 > 0$. Consequently,

$$|B_1\varphi(t)|^2 \leqslant e^{-\widetilde{C}_1 t}|B_1\varphi_0|^2 + \frac{1}{\widetilde{C}_1 C_1}(1-e^{-\widetilde{C}_1 t})\|\boldsymbol{f}\|^2_{V_T^*}. \tag{5.2}$$

Let

$$B_K = \{\varphi = (\boldsymbol{V}, 0, T)\in L^2(T\Omega)\times L^2(\Omega) \mid |\varphi|^2 \leqslant K\}, \tag{5.3}$$

where
$$K > \frac{1}{\widetilde{C}_1 C_1}\|\boldsymbol{f}\|^2_{V_T^*}.$$

If $|B_1\varphi_0|^2 \leqslant \frac{1}{\widetilde{C}_1 C_1}\|\boldsymbol{f}\|^2_{V_T^*}$, then

$$|B_1\varphi(t)|^2 \leqslant e^{-\widetilde{C}_1 t}\cdot\frac{1}{\widetilde{C}_1 C_1}\|\boldsymbol{f}\|^2_{V_T^*} + (1-e^{\widetilde{C}_1 t})\cdot\frac{1}{\widetilde{C}_1 C_1}\|\boldsymbol{f}\|^2_{V_T^*} \leqslant K,$$

i.e. $B_1\varphi(t)\in B_K$, $\forall t \geqslant 0$.

On the other hand, if $|B_1\varphi_0|^2 > \frac{1}{\widetilde{C}_1 C_1}\|\boldsymbol{f}\|^2_{V_T^*}$, then letting

$$t_0 = \frac{1}{\widetilde{C}_1}\ln\frac{|B_1\varphi_0|^2 - \frac{1}{\widetilde{C}_1 C_1}\|\boldsymbol{f}\|^2_{V_T^*}}{K - \frac{1}{\widetilde{C}_1 C_1}\|\boldsymbol{f}\|^2_{V_T^*}},$$

one has $|B_1\varphi(t)|^2 \leqslant K$ for all $t \geqslant t_0$. Thus, we have shown that:

Theorem 5.1. *Any solution φ of Problem 3.1 satisfies*

1) $\forall t \geqslant 0$, *$B_1\varphi(t)$ remains in B_K if* $|B_1\varphi_0|^2 \leqslant \frac{1}{\widetilde{C}_1 C_1}\|\boldsymbol{f}\|^2_{V_T^*}$;

2) *$B_1\varphi(t)\in B_K$ at least for $t \geqslant t_0$ if* $|B_1\varphi_0|^2 > \frac{1}{\widetilde{C}_1 C_1}\|\boldsymbol{f}\|^2_{V_T^*}$.

This theorem tells us that φ always goes into the bounded ball B_K at least for

t which is sufficiently large. So we call B_K the absorbing set of Problem 3.1. Since all the solution of Problem 3.1 will go into the ball, we also call it a global absorbing set. The above result also shows that the long-time behaviour of the system only depends on points in B_K, but not on points out of it. Of course, since the ω-equation does not contain ω explicitly, we cannot obtain any estimates about ω.

VI. Functional Invariant Set

In this section, we establish the existence of some functional invariant set. To this end, we assume that there is a subset Y of B_K such that all the solutions of Problem 3.1 with an initial value $B_1\varphi_0 \in Y$ satisfy the conditions given in Theorem 4.1 and

$$\sup_{t \geqslant 0}\{\|\varphi(t)\| \,|\, \varphi \text{ the solution with initial value } B_1\varphi_0 \in Y\} \leqslant M < \infty, \tag{6.1}$$

where M is a constant.

By Theorem 4.1 and its proof, $B_1\varphi$ is uniquely determined by its initial value $B_1\varphi_0$, and

$$B_1\varphi \in C([0,\infty), L^2(T\Omega) \times L^2(\Omega)). \tag{6.2}$$

So, let $S_t B_1\varphi_0 = B_1\varphi(t)$ then S_t is a continuous mapping of t. Therefore, one can define

$$X = \bigcap_{s>0} \overline{\bigcup_{t \geqslant s} S_t Y},$$

where the bar denotes the closure in $L^2(T\Omega) \times L^2(\Omega)$ and

$$S_t Y = \{S_t B_1\varphi_0 | \ \forall B_1\varphi_0 \in Y\}.$$

The main result in this section is

Theorem 6.1. *X is a functional invariant set, i.e. X is a bounded set in $L^2(T\Omega) \times L^2(\Omega)$ such that*

$$S_\theta X = X, \ \forall \theta \geqslant 0,$$

$$\sup_{B_1\varphi_0 \in Y} \inf_{x \in X} \{|S_t B_1\varphi_0 - x|\} \to 0, \ (t \to \infty). \tag{6.3}$$

Proof. That X is bounded in $L^2(T\Omega) \times L^2(\Omega)$ is obvious.

By the continuity of S_t, we have for any $\theta \geqslant 0$, $B_1\varphi_0 \in Y$,

$$S_\theta \overline{\{S_t B_1\varphi_0 | t \geqslant \tau\}} \subset \overline{\{S_{t+\theta} B_1\varphi_0 | t \geqslant \tau\}} \subset \overline{\{S_t B_1\varphi_0 | t \geqslant \tau + \theta\}}.$$

S_0, $S_\theta X \subset X$.

On the other hand, $\forall x \in X$, there is $t_j \to \infty$, $B_1\varphi_j \in Y$ such that

$$x = \lim_{j \to \infty} S_{t_j} B_1\varphi_j = \lim_{j \to \infty} S_\theta(S_{t_j - \theta} B_1\varphi_j).$$

By the assumption and Rellich compactness theorem, one may also assume that $S_{t_j-\theta} B_1\varphi_j \to z$. By the definition of $X, z \in X$, i.e. $X = S_\theta z$. In a word, $S_\theta X = X$.

If $\overline{\lim} \sup_{B_1\varphi_0 \in Y} \inf_{x \in X} \{|S_t B_1\varphi_0 - x|\} = 2\delta > 0$, then there are $t_j \to \infty, t_{j+1} > t_j, B_1\varphi_j \in Y$

such that

$$\inf_{x \in X}\{|S_{t_j}B_1\varphi_j - x|\} \geqslant \delta, \ j = 1, 2, \cdots. \tag{6.4}$$

So, we may assume that $S_{t_j}B_1\varphi_j \to x' \in X$, which contradicts with (6.4). Namely (6.3) holds true.

The above result tells us that as t increases, the system will tend to some functional invariant set, which represents the limiting state of the system. From the physical point of view, this means that the system is nonlinearly adjusted to exterior sources.

VII. Conclusions and Discussion

From the above discussion, we can see that if t is larger than some definite critical time t_0, the system will go into the bounded absorbing set B_K, more and more close to invariant set X, and the distance between them tends to zero, that is to say, the system is situated in the state of the attractor. For the real atmospheric system, if the change of the exterior sources as opposed to the monthly scale is a slow process, the long-range weather process studied by us is actually situated in the state of the attractor. Because the evolution of which the dissipative system shrank from the higher phase space to the lower attractor is in fact the process that lumped the degrees of freedom together, the dissipation expended a large amounts of faster small-scale motion and decreased in numbers of degrees of freedom determining the long-time behaviour of the atmospheric system. In the evolution many degrees of freedom changed to variables of no importance, and finally remained a few degrees of freedom which were chosen to describe the system state just including the degree of freedom which played a role at $t \to \infty$, then we could get a successful macroscopic description[6]. In other words, the long-range weather is different from the short and medium-range weather, we should set up an effective macroscopic description which represents the attractor state from the statistical analysis of theory and real observation data[7].

It is necessary to point out that this study is just the beginning of this topic; the dimensional estimation of attractor of the large-scale motion forcing by the exterior sources will be discussed further using the real data and theory.

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