INVERSION OF A NONLINEAR DYNAMICAL MODEL FROM THE OBSERVATION*

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ABSTRACT

A technique for inversion of a nonlinear dynamical model from the observations has been developed, and examined by using the Lorenz system. The results show that the Lorenz system can be accurately inversed by using the observation data set. This technique will be broadly used in establishing the nonlinear dynamical model from an actual data set.

Keywords: observation data, model inversion, nonlinear dynamical model.

I. INTRODUCTION

Recently, the nonlinear dynamics has been extensively developed. It has begun to pick up the information of dynamical system from time series, i.e. the calculation of fractional dimension and Lyapunov index[1]. However, these studies can only be used in understanding what the chaos systems are, what the periodic one is and how the stability of systems is. The dynamical behavior of the systems, such as the mechanism of bifurcation, catastrophe and chaos, cannot be studied in detail. One of the current diffculties is that we do not know the mathemtical models of such an actual system. So, how to establish the nonlinear dynamical model for the different systems is the most important theoretical task to be solved.

Although the dynamical models for many actual nonlinear systems are not understood, we have known a series of quite reliable particular solutions to these models, i.e. the observations which have been accumulated for many years. So, if the data set is regarded as a series of discrete solutions of a dynamical model and these discrete values are used to solve the inverse problem of numerical integration, a better nonlinear dynamical model can be developed.

II. BASIC INVERSION METHOD

It is assumed that the physical law of the temporal evolution of the system can be written as

$$\frac{dq_i}{dt} = f_i(q_1, q_2, \ldots, q_n), \ i = 1,2,\ldots,n, \tag{1}$$

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where f_i is the general nonlinear function. The number of state variables n can be determined according to the fractional dimension of the system.

Generally, we do not know the specific forms of the function f_i $(q_1, q_2, \ldots, q_n)$; but know a series of particular solutions of Eq. (1), i.e. $q_i^{(j\Delta t)}$ $(j=1, 2, \ldots, m$, m is the length of data series), so Eq. (1) can be written in the difference form

$$\frac{q_i^{(j+1)\Delta t} - q_i^{(j-1)\Delta t}}{2\Delta t} = f_i(q_1^{j\Delta t}, q_2^{j\Delta t}, \ldots, q_n^{j\Delta t}), \; j = 2, \ldots, m-1. \tag{2}$$

Further, f_i is assumed to be a nonlinear function according to the physical characteristic of system and then we determine the specific form and the parameters by using the inversion method.

It is suggested that there are $\boldsymbol{G}_k$ terms and $\boldsymbol{P}_k$ parameters in $f_i(q_1, q_2, \ldots, q_n)$, i.e.

$$f_i(q_1, q_2, \ldots, q_n) = \sum_{k=1}^{K} \boldsymbol{G}_k \boldsymbol{P}_k$$

and that the observation data can form M equations $(M = m - 2)$. It can be written in a matrix form:

$$\boldsymbol{D} = \boldsymbol{G}\boldsymbol{P}, \tag{3}$$

where

$$\boldsymbol{D} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_M \end{pmatrix} = \begin{pmatrix} \dfrac{q_i^{3\Delta t} - q_i^{\Delta t}}{2\Delta t} \\ \dfrac{q_i^{4\Delta t} - q_i^{2\Delta t}}{2\Delta t} \\ \vdots \\ \dfrac{q_i^{m\Delta t} - q_i^{(m-2)\Delta t}}{2\Delta t}, \end{pmatrix},$$

$$\boldsymbol{G} = (G_1, G_2, \cdots, G_K) = \begin{pmatrix} G_{11} & G_{12} \cdots G_{1K} \\ G_{21} & G_{22} \cdots G_{2K} \\ \vdots & \vdots \quad \vdots \\ G_{M1} & G_{M2} \cdots G_{MK} \end{pmatrix}, \quad \boldsymbol{P} = \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_K \end{pmatrix}.$$

$\boldsymbol{G}$ is the $M \times K$ matrix obtained from the observed data and the vector $\boldsymbol{D}$ in Eq. (3) is also already known. So the problem is how to determine the vector $\boldsymbol{P}$. Because it is a linear system for $\boldsymbol{P}$, the ordinary least square method can be used to determine parameter $\boldsymbol{P}$, i.e. take the sum of deviation square as minimum, i.e.

$$\boldsymbol{S} = (\boldsymbol{D} - \boldsymbol{G}\boldsymbol{P})^T(\boldsymbol{D} - \boldsymbol{G}\boldsymbol{P}), \tag{4}$$

where T means transpose. According to the criterion of the least square method, we have the following regular equation:

$$\boldsymbol{G}^T\boldsymbol{G}\boldsymbol{P} = \boldsymbol{G}^T\boldsymbol{D}. \tag{5}$$

If the $\boldsymbol{G}^T\boldsymbol{G}$ is nonsingular matrix, we have

$$\boldsymbol{P} = (\boldsymbol{G}^T\boldsymbol{G})^{-1}\boldsymbol{G}^T\boldsymbol{D}, \tag{6}$$

but the matrix $\boldsymbol{G}$ in Eq. (3) is often a singular matrix, or approximates to a singular one. When it approximates to a singular one, the solutions are extremely sensitive to the errors. Unfortunately, the matrix $\boldsymbol{G}$ itself cannot be known accurately and has large error. The inverse theory can be used to overcome this difficulty[2,3].

First, the real symmetric $K\times K$ matrix $\boldsymbol{G}^T\boldsymbol{G}$ is calculated, in which all the eigenvalue are real, and there are K eigenvectors which are linearly independent and orthogonal in the following form:

$$|\lambda_1| \geqslant |\lambda_2| \geqslant ,\ldots, |\lambda_K|.$$

It is assumed that there are L non-zero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_L$ and $K-L$ eigenvalues are zero (or approximately zero). The matrix $\boldsymbol{U}_L$, which corresponds to the standardizing eigenvectors of L non-zero eigenvalues, is written as

$$\boldsymbol{U}_L = \begin{pmatrix} U_{11} & U_{12}\cdots U_{1L} \\ U_{21} & U_{22}\cdots U_{2L} \\ \vdots & \vdots \quad\quad \vdots \\ U_{K1} & U_{K2}\cdots U_{KL} \end{pmatrix},$$

where $U_i = (U_{1i}, U_{2i}, \ldots, U_{Ki})$, $i = 1, 2, \ldots, L$, is the eigenvectors corresponding to λ_i. Then, we calculate V_i:

$$V_i = \frac{1}{\lambda_i}\boldsymbol{G}U_i = (V_{1i}, V_{2i}, \ldots, V_{Mi})^T$$

and have

$$\boldsymbol{V}_L = \begin{pmatrix} V_{11} & V_{12}\cdots V_{1L} \\ V_{21} & V_{22}\cdots V_{2L} \\ \vdots & \vdots \\ V_{M1} & V_{M2} \quad V_{ML} \end{pmatrix}.$$

Let the diagonal matrix consist of eigenvalues as L

$$\boldsymbol{\Lambda}_L = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \lambda_L \end{pmatrix}$$

and take $\boldsymbol{H} = \boldsymbol{U}_L\boldsymbol{\Lambda}_L^{-1}\boldsymbol{V}^T$, obviously

$$\boldsymbol{P} = \boldsymbol{H}\boldsymbol{D}. \tag{7}$$

So, the parameter $\boldsymbol{P}$ can be determined by using Eq. (7).

Finally, the inversed equations are obtained through cutting out the independent terms which make no significant contribution to the system evolution. In order to raise the accuracy of inversing model, the equations in which the independent terms have been cut out can be inversed again by using the original data set. In the next section the Lorenz system will be taken as an example to show how to create the nonlinear dynamical model from the observation data set.

III. Inversion of Lorenz System

As we know, the Lorenz system can be written as

$$\frac{dX}{dt} = \sigma Y - \sigma X, \tag{8}$$

$$\frac{dY}{dt} = rX - Y - XZ, \tag{9}$$

$$\frac{dZ}{dt} = XY - bZ, \tag{10}$$

where $\sigma = 10$, $r = 28$, $b = 8/3$.

First, Eqs. (8)—(10) are integrated by using the Rugo-Kuta method and we regard the every-step results of integration as an ideal observed record. So, an ideal time series of the variables X, Y, Z can be obtained. And here we take the integration results from 2000th to 3000th steps as the time series. The integration initial was taken as (0,1,0) and the time increment as 0.005. Further, we regard the inversed system as

$$\frac{dX}{dt} = a_1X + a_2Y + a_3Z + a_4X^2 + a_5Y^2 + a_6Z^2 + a_7XY + a_8XZ + a_9YZ, \tag{11}$$

$$\frac{dY}{dt} = b_1X + b_2Y + b_3Z + b_4X^2 + b_5Y^2 + b_6Z^2 + b_7XY + b_8XZ + b_9YZ, \tag{12}$$

$$\frac{dZ}{dt} = c_1X + c_2Y + c_3Z + c_4X^2 + c_5Y^2 + c_6Z^2 + c_7XY + c_8XZ + c_9YZ. \tag{13}$$

It is obvious that there are not only the terms included in the original Lorenz system but also other independent terms. A focal point of this example is how to assess the coefficients of each term and cut out the independent terms. The inversion

Table 1

The Inversion Coefficients for the Terms

Coefficient	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
Actual Value	−10	10	0	0	0	0	0	0	0
Inversion	−10.20	10.07	-3.91×10^{-4}	-2.79×10^{-4}	-6.32×10^{-4}	2.67×10^{-5}	2.82×10^{-4}	5.69×10^{-3}	-1.13×10^{-3}
Coefficient	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
Actual Value	28	−1	0	0	0	0	0	−1	0
Inversion	28.16	−1.19	-6.48×10^{-4}	3.05×10^{-5}	-9.79×10^{-5}	4.55×10^{-6}	8.9×10^{-7}	−1.01	9.35×10^{-3}
Coefficient	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
Actual Value	0	0	−2.66	0	0	0	1	0	0
Inversion	1.78×10^{-3}	-7.72×10^{-4}	−2.66	9.41×10^{-3}	-7.77×10^{-3}	-3.99×10^{-4}	1.00	-7.69×10^{-5}	1.41×10^{-5}

results of the coefficients for each terms in the system are shown in Table 1. It can be found that the coefficients of the original system agree well with the actual value, but the coefficients of the independent one are three orders of magnitude less than that of the original system and approximate to zero. The difference between them is obvious.

To quantitatively compare the relative contribution of each term with the evolution of system, the relative variance of each term was computed. The formula is

$$R_i = \frac{1}{m}\sum_{j=1}^{m}\left[T_i^2\Big/\left(\sum_{i=1}^{9} T_i^2\right)\right], \qquad (14)$$

where $m = 1000$ is the length of data series, $T_i = a_1X, \dots, a_9YZ$, the terms in the equations. The R_i values of the three equations are shown in Table 2. It is shown that the terms of the original system contribute greatly to the variance while the contribution of independent one is nearly zero, the sum of them is below 1%.

Table 2

The Relative Variance for the Terms

Terms	a_1X	a_2Y	a_3Z	a_4X^2	a_5Y^2	a_6Z^2	a_7XY	a_8XZ	a_9YZ
R_i	0.485	0.514	1.138×10^{-9}	3.987×10^{-8}	2.35×10^{-9}	9.796×10^{-10}	4.76×10^{-9}	2.169×10^{-5}	8.949×10^{-8}
Terms	b_1X	b_2Y	b_3Z	b_4X^2	b_5Y^2	b_6Z^2	b_7XY	b_8XZ	b_9YZ
R_i	0.862	0.062	1.189×10^{-9}	5.765×10^{-11}	6.55×10^{-8}	3.459×10^{-12}	5.526×10^{-14}	0.076	2.27×10^{-5}
Terms	c_1X	c_2Y	c_3Z	c_4X^2	c_5Y^2	c_6Z^2	c_7XY	c_8XZ	c_9YZ
R_i	1.375×10^{-7}	1.415×10^{-7}	0.351	4.0×10^{-4}	2.02×10^{-3}	1.25×10^{-6}	0.647	2.48×10^{-8}	2.6×10^{-10}

Table 3

The Inversion Coefficients

Coefficient	a_1	a_2	b_1	b_2	b_8	c_3	c_7
Actual Value	−10	10	28	−1	−1	−2.66	1
Inversion	−9.98	9.92	27.7	−0.89	−0.99	−2.66	1

So, the independent terms are easily removed through comparing Table 1 with Table 2. The final inversed equations are

$$\frac{dX}{dt} = a_1X + a_2Y, \qquad (15)$$

$$\frac{dY}{dt} = b_1X + b_2Y + b_8XZ, \qquad (16)$$

$$\frac{dZ}{dt} = c_3Z + c_7XY. \qquad (17)$$

Comparing Eqs. (15)—(17) with Eqs. (11)—(13), the inversed equations are in complete agreement with the original equations. To improve the inversion accuracy Eqs. (15)—(17) can be inversed again by using the data series and the coefficients inversed are shown in Table 3. In comparison with those of Table 1, the coefficients shown in Table 3 approximate to the actual values to a great extent. It demonstrates that the complex nonlinear chaos dynamical model like the Lorenz system can be accurately inversed from the actual 'observation'. Though the observational data used here are replaced by integrating values, both of them are equal to each other in value if the observational data are accurate enough.

IV. Conclusions and Discussions

The results mentioned above show that the nonlinear dynamical model can be inversed accurately by using the observational data. It will provide a new way to study nonlinear dynamics, especially to design a dynamical model for the actual problem, which cannot be known accurately. However, it should be pointed out that there are some specific difficulties when it is applied to the actual problem.

(1) Due to the obvious 'noise' contained in the original data set, it is advantageous to filter it before using it.

(2) Many experiment results show that the longer the length of data series, the higher the accuracy of the inversion. However, the length of data series for many specific problems are not long enough. In order to guarantee the inversion accuracy, the following approach of repeating inversion can be applied,

$$\boldsymbol{D} - \boldsymbol{G}\tilde{\boldsymbol{P}}_n = \boldsymbol{G}\Delta\boldsymbol{P}_n, \tag{18}$$

where $\tilde{\boldsymbol{P}}_n$ is the coefficient of last time, and $\tilde{\boldsymbol{P}}_n = \tilde{\boldsymbol{P}}_{n-1} + \Delta\boldsymbol{P}_{n-1}$. Eq. (18) can be integrated again and again until the accuracy reaches the required value, i.e. $\Delta\boldsymbol{P} \to 0$. So the better results can also be obtained by using this approach when data series are short.

(3) The variable q_i is required to statisfy Eq. (1), i.e. time "t" does not appear at the right of equation, it must be a self-governing system.

(4) The observational data set should be standardized (i.e. dimensionless) and a suitable characteristic time should be selected for the agreement of both the interval period of observation and the time increment of difference. If the time interval between the observations is too long, the interpolation between observations at two times can be used as rough approximation.

(5) For the actual problem, the diagnosis analysis should be first performed so as to determine what variable should be selected as q_i.

Considering the preliminary character of this work, it should be improved further.

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